Amenability and convex Ramsey theory in the metric setting

Adriane Kaïchouh

Université Lyon 1

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Notions of topological dynamics

Let G be a topological group.

Definition

The group G is extremely amenable if every continuous action of G on a compact space admits a fixed point.

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The group G is amenable if every continuous action of G on a compact space admits an invariant Borel probability measure.

Polish groups as automorphism groups

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  A metric Fraïssé class (Ben Yaacov) is the class of all finite metric structures that embed into a given metric Fraïssé structure.

Characterization in the classical setting

Let M a countable Fraïssé structure and let $\mathcal{K}$ be the associated Fraïssé class.

Theorem (Moore, 2011)

Aut(M) is amenable if and only if $\mathcal{K}$ has the convex Ramsey property.

The Ramsey property and extreme amenability

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Theorem (Kechris - Pestov - Todorčević, 2005)

$\mathrm{Aut}(M)$ is extremely amenable if and only if $\mathcal{K}$ has the Ramsey property.

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$\beta_1, ..., \beta_n \in \mathrm{Emb}(B, C)$ such that for every $\alpha, \alpha' \in \mathrm{Emb}(A, B)$,

$$\left| \sum_{i=1}^{n} \lambda_i f(\beta_i \circ \alpha) - \sum_{i=1}^{n} \lambda_i f(\beta_i \circ \alpha') \right| < \epsilon.$$

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- For the implication [convex Ramsey property $\Rightarrow$ amenability], because of these regularity restrictions, Moore's proof cannot adapt.
- Main tool: Lipschitz functions are dense in uniformly continuous ones for the topology of uniform convergence.

Amenability is a G_δ condition

Melleray and Tsankov proved, in 2011, that extreme amenability is a G_δ condition (in the following sense), and the same is true of amenability.

Theorem (K., 2013)

Let Γ be a countable group and G be a Polish group. Then the set

$$\{\pi \in \mathrm{Hom}(\Gamma, G) : \pi(\Gamma) \text{ is amenable for the topology induced by } G\}$$

is G_δ in $\mathrm{Hom}(\Gamma, G)$.

Switching quantifiers: $\exists\forall \rightsquigarrow \forall\exists$

Proposition

Let Γ be a discrete group. Then the following are equivalent;

- The group Γ is amenable.
- There exists a finitely additive measure m on Γ such that for every subset $E \subseteq \Gamma$ and for all $\gamma \in \Gamma$, one has $m(\gamma E) = m(E)$.

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Theorem (K., 2013)

Let G be a Polish group. Then the following are equivalent.

- The group G is amenable.
- For every right uniformly continuous bounded function $f : G \to [0,1]$, there exists a positive linear form Λ of norm 1 on $\mathrm{RUCB}(G,[0,1])$ such that for all $g \in G$, one has $\Lambda(g \cdot f) = \Lambda(f)$.

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The same is true for extreme amenability with multiplicative linear forms.